Linear-Time Recognition of Map Graphs with Outerplanar Witness

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Abstract

Map graphs generalize planar graphs and were introduced by Chen, Grigni and Papadimitriou [STOC 1998, J.ACM 2002]. They showed that the problem of recognizing map graphs is in \textit{NP} by proving the existence of a planar witness graph \textit{W}. Shortly after, Thorup [FOCS 1998] published a polynomial-time recognition algorithm for map graphs. However, the run time of this algorithm is estimated to be \(\Omega(n^{120})\) for \(n\)-vertex graphs, and a full description of its details remains unpublished.

We give a new and purely combinatorial algorithm that decides whether a graph \(G\) is a map graph having an outerplanar witness \(W\). This is a step towards a first combinatorial recognition algorithm for general map graphs. The algorithm runs in time and space \(O(n + m)\). In contrast to Thorup’s approach, it computes the witness graph \(W\) in the affirmative case.

Keywords. Planar graphs, map graphs, certifying algorithms.

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1 Introduction

Consider the adjacency graph of the states of the USA, where two states are adjacent if their borders intersect. Since Arizona, Colorado, New Mexico and Utah meet pairwise at a single common point, the adjacency graph will not be planar; however, it will be a map graph. In (much) more detail, a map of a graph \(G = (V, E)\) is a function \(M\) that maps each vertex \(v \in V\) to a disc homeomorph \(M(v)\) on the sphere (the states) such that, for any two distinct vertices \(v, w \in V\), the interiors of \(M(v)\) and \(M(w)\) are disjoint, and \(v\) and \(w\) are adjacent in \(G\) if and only if the boundaries of \(M(v)\) and \(M(w)\) intersect. A graph \(G\) is a map graph if a map of \(G\) exists. The points of the sphere that are not covered by \(M\) fall into open connected regions; the closure of each such region is a hole of \(M\).

By definition, map graphs contain and exceed the class of planar graphs. They have applications in graph drawing, circuit board design and topological inference problems [4]. Chen, Grigni and Papadimitriou [2] characterized map graphs as the half-squares of sufficiently small planar bipartite graphs called witnesses (we give precise definitions for both terms in the next chapter). This result allows, similar to Kuratowski’s Theorem for planar graphs, to use purely combinatorial arguments for an object that has been originally defined by topological properties. Since such witnesses can always be chosen small in size (\(O(n)\) vertices for map graphs on \(n\) vertices), the recognition problem for map graphs is in \textit{NP}. In 1998, Chen et al. therefore raised the question whether recognizing map graphs is in \textit{P}.

This problem was resolved shortly after by Thorup [20], whose solution is based on a carefully designed topological treatment. However, a full version of the extended abstract [20] has, to the best of our knowledge, not yet appeared. The algorithm is complicated; its run time is not given explicitly, but estimated to be at least \(O(n^{120})\). Moreover, driven by topological arguments, the algorithm does not produce a witness if the graph is indeed a map graph, although a combinatorial description of this witness is at hand. In this view, an important question left open is whether there is a polynomial-time certifying

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algorithm in the sense of McConnell et al. [16], where a good candidate for a certificate would be the witness mentioned above.

Our Contribution. We give a purely combinatorial recognition algorithm for map graphs that have an outerplanar witness (rather than a planar witness). Map graphs with an outerplanar witness are general enough that they can have unbounded treewidth; in particular, cliques of any size may belong to this class of graphs.

**Theorem 1.** There is an algorithm that decides in time $O(n + m)$ whether $G$ is a map graph with an outerplanar witness $W$, and if so, outputs $W$ and a map of $G$.

Our algorithm runs in time and space $O(n + m)$ and is certifying. This is the first non-trivial step towards a combinatorial and efficient recognition algorithm for general map graphs. Although the restriction to outerplanar witnesses is somewhat specific compared to the general case of planar witnesses, we will show structural properties for certain classes beyond (e.g. for $K_{2,k}$-free witnesses, and for graphs with small separators), that might be important for solving the general case.

We remark that the main algorithmic task is to compute a witness $W$, or to decide that none exists. Creating a map from $W$ is a simple task that can be accomplished in linear time [2]. The crucial part of computing a witness $W$ is that we know only a subset of the vertices of $W$; we need to do non-trivial algorithmic work in order to compute the remaining vertices of $W$. This is the reason why recognizing graphs that are half-squares of planar graphs is more challenging than recognizing graphs that are squares of planar graphs [14].

**Related Work.** By definition, planar graphs are an important subclass of map graphs, and planar graphs have been known since the 1970s to be recognizable in time $O(n)$ [12]. Nowadays, several other linear-time algorithms for planar graph recognition exist [18], and it is natural to ask whether they can be generalized to the much wider class of map graphs. Let a $d$-map graph be a map graph that has a witness in which every intersection point has at most $d$ neighbors (states). The planar graphs are exactly the map graphs for which at most three states meet at each single point; thus, by the well-known linear-time recognition algorithms for planar graphs, 3-map graphs can be recognized in $O(n)$ time.

An intricate cubic-time recognition algorithm for a subclass of 4-map graphs was given by Chen et al. [3]; here, the 4-map graphs are required to be hole-free, meaning that there is at most one connected region of the plane that is not covered by states or borders. However, even efficiently recognizing general 4-map graphs in polynomial time remains an open problem; Thorup’s algorithm does not necessarily give an embedding minimizing the maximum degree of the intersection points, so it cannot be used to recognize 4-map graphs.

Another motivation for $d$-map graphs is the study of 1-planar graphs, which are the graphs that can be embedded in the plane such that each edge crosses at most one edge. Recognizing 1-planar graphs is NP-complete [10,14]. Chen et al. [3] observe that triconnected hole-free 4-map graphs form a special class of 1-planar graphs, which is hence efficiently recognizable. It would be interesting to know where exactly the recognition problem becomes NP-complete between these two graph classes.

Further interest stems from the parameterized complexity community: Generalizing earlier algorithms for problems on planar graphs, Demaine et al. [5] gave fixed-parameter algorithms for combinatorial optimization problems such as minimum dominating set in map graphs. Fomin et al. [9] gave PTAS’s for optimization problems on map graphs; they later improved these to EPTAS’s [8].

# 2 Preliminaries

All graphs considered in this paper are finite, simple, and undirected. For a graph $G$, let $V(G)$ and $E(G)$ denote its vertex set and edge set, and let $n := |V(G)|$ and $m := |E(G)|$. For a vertex $v \in V(G)$, let $N_G(v)$ be the set of neighbors of $v$ in $G$. For a subset $V' \subseteq V(G)$, let $G[V']$ denote the subgraph of $G$ induced by $V'$. For a graph $G$, its square $G^2$ is the graph on vertex set $V(G)$ in which two vertices are adjacent if their distance in $G$ is at most two.

**Witnesses.** A witness of a map graph $G = (V, E)$ is a bipartite planar graph $W = (V \sqcup I, E_W)$ with $E_W \subseteq V \times I$ such that $W^2[V] = G$. The graph $W^2[V]$ is also called the half-square of $W$, as it is the square of $W$ restricted to the side $V$ of the bipartition. The vertices in $I$ are called intersection
If a witness $W$ is a tree witness if it is a tree; analogously, outerplanar witnesses are outerplanar and the usual witnesses, which are planar, are sometimes called planar witnesses.

**Proposition 1** ([2]). A graph $G$ is a map graph if and only if it has a witness. If so, there is a witness with at most $3n - 6$ intersection points.

A direct consequence of this result is that the recognition problem for map graphs is in \textbf{NP}. Let $G$ be a map graph with witness $W = (V \cup I, E_W)$. Throughout this paper, we assume, without loss of generality, that every intersection point in $I$ has degree at least two.

Let $G$ be a map graph with witness $W$ and let $P = v_1, v_2, \ldots, v_k$ be a path in $G$. A path $P_W$ in $W$ corresponds to $P$ if $P_W = v_1, x_1, v_2, x_2, \ldots, x_{k-1}, v_k$ such that $x_i$ is an intersection point that is adjacent to $v_i$ and $v_{i+1}$, for $i = 1, \ldots, k - 1$. Observe that any path $P$ in $G$ has some corresponding path in $W$, and any induced path $P$ in $G$ has a corresponding induced path $P'$ in $W$, as every chord of $P'$ in the bipartite witness $W$ would join an intersection point with a real vertex and therefore generate a chord of $P$. More generally, for a subgraph of $G$ that is induced by some vertex subset $U \subseteq V(G)$, we specify the corresponding part in a witness of $G$.

**Definition 1.** Let $G$ be a map graph with witness $W$ and let $U \subseteq V(G)$. A vertex $w \in W$ is touched by $U$ if either $w \in U$ or $w$ is an intersection point with at least two neighbors in $U$. The touched set $T(U)$ of $U$ is the set of all vertices in $W$ touched by $U$. The touched subgraph of $U$ is $W[T(U)]$.

By using half-squares, we can get back from an induced subgraph $W[T(U)]$ of $W$ for some $U \subseteq V(G)$ to the original subgraph $W^2[U]$ in $G$. Clearly, $W[T(U)]$ witnesses $W^2[U]$. We will often use the following observation.

**Observation 1.** For every $U \subseteq V$, $W[T(U)]$ is a witness of $G[U]$. Moreover, $G[U]$ is connected if and only if $W[T(U)]$ is connected.

**Outerplanar Graphs.** The following characterizations of planar and outerplanar graphs in terms of forbidden minors are well-known.

**Proposition 2** (Wagner [21]). A graph is planar if and only if it neither contains a $K_5$-minor nor a $K_{3,3}$-minor.

**Proposition 3.** A graph is outerplanar if and only if it neither contains a $K_4$-minor nor a $K_{2,3}$-minor.

**Proposition 4** (Syslo [19]). A triangle-free graph is outerplanar if and only if it does not contain a $K_{2,3}$-minor.

**Connectivity and SPQR-trees.** A graph is connected if every two of its vertices are connected by a path; the maximal connected subgraphs of $G$ are called components of $G$. A separator $S$ of a graph $G$ is a subset of $V$ such that $G - S$ has more components than $G$. For an integer $c \in \mathbb{N}$, a connected graph is $c$-connected if it either has at most $c$ vertices or removing any set of less than $c$ vertices leaves a connected subgraph. A 2-connected resp. 3-connected graph is sometimes called biconnected resp. triconnected.

For a graph $G$, an SPQR-tree is a tree $T$ for which each node $x \in V(T)$ has an associated multigraph $G_x$, called skeleton of $x$, and one of the following four types:

- **S-node:** then $G_x$ is a cycle on at least three vertices.
- **P-node:** then $G_x$ is a multigraph with two vertices and at least three edges.
- **Q-node:** then $G_x$ is a multigraph with two vertices and two parallel edges.
- **R-node:** then $G_x$ is a 3-connected graph.

Each edge $xy$ between two nodes of $T$ is associated with two directed virtual edges, one in $G_x$ and one in $G_y$. Each edge in $G_x$ can be virtual for at most one edge of $T$. All edges of $S$-, $P$- and $R$-nodes are virtual for some edge of $T$, and we simply call them virtual edges. An edge that is not virtual for any edge of $T$ is real. Only skeletons of $Q$-nodes contain real edges and every $Q$-node skeleton contains exactly one real edge.
An SPQR-tree $T$ represents a biconnected graph $G_T$, formed as follows. Whenever an edge $xy \in E(T)$ associates the virtual edge of $G_x$ with the virtual edge of $G_y$, form a larger graph as the 2-clique-sum of $G_x$ and $G_y$. We identify the endpoints of the virtual edge of $G_x$ with that of $G_y$, and then delete the resulting edge. Applying this step to each edge of $T$ (in any order) produces the graph $G_T$.

We assume throughout that $T$ is minimal, which implies that its $S$- and $P$-nodes are pairwise non-adjacent. Under this assumption, $T$ is uniquely determined from $G$. The graphs $G_x$ associated with the nodes of $T$ are called the triconnected components of $G$.

While the above definition coincides with the classical definition of SPQR-trees, it is often more convenient to omit the Q-nodes from the tree as they carry little information. To this end, we simply remove each Q-node and replace the corresponding virtual edge in the skeleton of the neighboring node by a real edge. In the following we will use this modified version of SPQR-trees.

## 3 Reduction along Small Separators

Clearly, every separator $S$ of any witness $W$ of a map graph $G$ that contains only vertices of $V(G)$ and for which at least two components of $W - S$ contain vertices in $V(G)$ is also a separator in $G$, as no edge that is generated by the half-square can cross $S$.

### Lemma 1

Let $G$ be a map graph with witness $W$, let $S \subseteq V(G)$ and let $C$ be the family of vertex sets of the components of $G - S$. Then $C \mapsto W[T(C)]$ is a bijection to the components of $W - S$ that contain a vertex of $V(G)$. In particular, every separator $S$ of $G$ is a separator of $W$ and, conversely, every separator $S \subseteq V(G)$ of $W$ such that at least two components of $W - S$ contain a vertex of $V$ is a separator of $G$.

**Proof.** If $S$ is not a separator of $G$ or of $W$, the statement follows from Observation [1]. Hence, assume that $S$ separates both $G$ and $W$. Let $A$ and $B$ be the vertex sets of two arbitrary components of $G - S$. By Observation [1], the touched subgraphs $W[T(A)]$ and $W[T(B)]$ are connected in $W - S$. Suppose, for sake of contradiction, that some vertices $a \in A$ and $b \in B$ are contained in the same component of $W - S$. Then $W - S$ contains a shortest path from $a$ to $b$, whose original subgraph in $G$ must be a path from $a$ to $b$ on the same real vertices, i.e., disjoint from $S$. This contradicts that $A$ and $B$ are different components of $G - S$; hence, the components of $G - S$ partition $V$ in exactly the same way as the components of $W - S$. In order to show that every $W[T(C)]$ is a component of $W - S$, it remains to prove that no intersection point is contained in two touched subgraphs $W[T(A)]$ and $W[T(B)]$. However, in that case, $A$ and $B$ would be connected in $G - S$.

### Lemma 2

A map graph $G$ has a planar (outerplanar, tree) witness if and only if all of its biconnected components have planar (outerplanar, tree) witnesses, respectively.

**Proof.** Assume $G$ has a planar (outerplanar, tree) witness $W$ and let $C$ be the vertex set of any biconnected component of $G$. Then $W[T(C)]$ is a planar (outerplanar, tree) witness for $G[C]$ by Observation [1] as trees, planar and outerplanar graphs are closed under taking induced connected subgraphs.

If, on the other hand, each biconnected component of $G$ has a planar (outerplanar, tree) witness, we can identify these witnesses along the cutvertices of $G$, obtaining a planar (outerplanar, tree) witness of $G$.

We will thus assume that $G$ is biconnected throughout the paper. Lemma [2] can be generalized to separators of size two as follows (a similar generalization exists for separators of size three). Consider a separator $S = \{u, v\}$ of size two in a biconnected graph. An $S$-bridge is either the edge $uv$, or the graph that is obtained from a component $C$ of $G - S$ by adding the edges of $G$ that join $C$ with $S$, as well as their endpoints.

### Lemma 3

Let $G$ be a biconnected map graph that is not triconnected, and let $S = \{u, v\}$ be a separator of $G$. If $uv$ is an edge of $G$, let $G' = G[C \cup S]$ for some component $C$ of $G - S$, otherwise let $G'$ be the graph obtained from $G$ by contracting some $S$-bridge $B$ of $G$ to a single edge. Then $G'$ is a map graph, and any witness of $G$ contains some witness of $G'$ as a minor.

**Proof.** Let $W$ be an arbitrary witness of $G$. If $uv$ is an edge of $G$, then $G'$ is an induced subgraph of $G$ and therefore a map graph. In this case, the touched subgraph $W[T[V(G')]])$ is an induced subgraph of $W$ and thus a minor of $W$, which gives the claim.
Now assume that \( G \) does not contain the edge \( S \). Then \( B \) contains a shortest path \( P \) connecting \( u \) and \( v \). Obtain the graph \( G'' \) from \( G \) by removing all vertices of \( V(B) \setminus V(P) \). Then \( G' \) can be obtained from \( G'' \) by contracting the path \( P \) (all whose interior vertices have degree 2) to a single edge. Since \( G'' \) is an induced subgraph of \( G \), \( W \) contains a witness \( W'' \) of \( G'' \) as an induced subgraph, as shown above. Since \( P \) is shortest and \( uv \notin G \), \( P \) is induced in \( G'' \); thus, as argued before Definition \([1]\) the witness \( W'' \) contains a path \( P'' \) realizing the path \( P \). The internal vertices of \( P'' \) all have degree 2, and so contracting this path \( P'' \) to a path of length 2 yields a corresponding witness \( W' \) for \( G' \). As we used only vertex deletions and contractions, \( W' \) is a minor of \( W \).

Lemma \([4]\) will allow us to reduce along separators of size two.

4 Map Graphs with a Tree Witness

We characterize the map graphs that admit a tree witness. The characterization implies immediately a linear-time recognition algorithm for such graphs.

**Lemma 4.** A biconnected map graph has a tree witness if and only if it is a clique.

**Proof.** Clearly a clique has a tree witness, namely a star. Conversely, assume that \( G \) is a biconnected graph with tree witness \( W \) and assume that \( G \) is not a clique. Then \( W \) is not a star, and it hence has two adjacent non-leaf vertices \( u \) and \( v \). Since intersection points are pairwise non-adjacent, one of them, without loss of generality \( v \), is not an intersection point. Then \( v \) is a cutvertex in \( W \) and, by Lemma \([1]\), a cutvertex in \( G \). This contradicts the assumption that \( G \) is biconnected.

Lemma \([2]\) and Lemma \([4]\) immediately imply the following characterization of map graphs with a tree witness.

**Theorem 2.** A map graph has a tree witness if and only if each of its biconnected components is a clique.

**Corollary 1.** Map graphs with a tree witness can be recognized in \( O(n + m) \) time.

5 Map Graphs with an Outerplanar Witness

In this section we study the problem of recognizing map graphs with an outerplanar witness. Due to Lemma \([2]\) we can assume that the input graph \( G \) is biconnected. As bipartite planar graphs are triangle-free, we know with Proposition \([3]\) that \( G \) has an outerplanar witness if and only if \( G \) has a \( K_{2,3} \)-minor free witness. Thus, all of the following proofs extend to the seemingly wider class of witnesses that are \( K_{2,3} \)-minor free.

The next result states that triconnected map graphs \( G \) have witnesses with a very simple structure. For \( k \geq 2 \), a set of paths \( \Pi_1, \ldots, \Pi_k \) in a witness \( W = (V(G) \cup I, E_W) \) of \( G \) is internally \( V(G) \)-disjoint if no two paths \( \Pi_i \) and \( \Pi_j \) share an internal vertex in \( V(G) \).

**Lemma 5.** For \( k \geq 3 \), a \( k \)-connected map graph \( G \) has a \( K_{2,k} \)-minor free witness if and only if it is a clique.

**Proof.** If \( G \) is a clique, it has a tree-witness with an intersection point as inner vertex and \( n \) leaves, and this witness is \( K_{2,k} \)-minor free. If \( G \) is not a clique, two vertices, say \( u, v \in V(G) \), are not adjacent; let \( W \) be a witness of \( G \). Since \( G \) is \( k \)-connected, \( G \) contains \( k \) internally vertex-disjoint paths from \( u \) to \( v \). Let \( P_1, \ldots, P_k \) denote such internally vertex-disjoint \( u \)-paths of minimum total length; in particular, each of the \( P_i \) is an induced path. Denote by \( u_i \) the neighbor of \( u \) in \( P_i \) for \( i = 1, \ldots, k \); see Fig. 1a for an example for \( k = 3 \). Let \( \Pi_i \) denote a path in \( W \) corresponding to \( P_i \) for \( i = 1, \ldots, k \). Clearly the \( \Pi_i \) are internally \( V(G) \)-disjoint and each of them is an induced path. For a path \( \Pi \) containing vertices \( a \) and \( b \), let \( \Pi[a, b] \) denote the subpath of \( \Pi \) from \( a \) to \( b \). Let \( A = \bigcup_{i=1}^k V(\Pi_i \backslash \{u, v\}) \cup \{u_1, \ldots, u_k\} \) and \( B = \bigcup_{i=1}^k V(\Pi_i \backslash \{u_i, v\}) \cup \{u_1, \ldots, u_k\} \); see Fig. 1b. Note that \( A \) consists exactly of \( u \) and the neighbors of \( u \) on the paths \( \Pi_i \).

We claim that

(i) \( W[A] \) and \( W[B] \) are connected,
(ii) \( A \cap B = \emptyset \), and

(iii) each vertex \( u_i \) has a neighbor in \( A \) and a neighbor in \( B \).

Assume that the claims hold and consider the graph \( W' := W[A \cup B \cup \{u_1, \ldots, u_k\}] \). Since \( W[A] \subseteq W' \) and \( W[B'] \subseteq W' \) are connected (Claim (i)) and disjoint (Claim (ii)), we can contract these subgraphs into distinct vertices \( v_A \) and \( v_B \), respectively. By Claim (iii), it follows that each of the \( u_i \) is adjacent to both \( v_A \) and \( v_B \). Omitting a possible edge \( v_Av_B \) yields a \( K_{2,k} \)-minor in \( W \).

We now prove the claims. Statements (i) and (iii) follow immediately from the definitions of \( A \) and \( B \) via the paths \( \Pi_i \). For (ii), assume that \( A \cap B \neq \emptyset \) and let \( x \in A \cap B \). Since the paths \( \Pi_1, \ldots, \Pi_k \) are internally \( V(G) \)-disjoint and each of them is induced, \( x \) must be an intersection point. Since \( x \in A \), \( x \) is adjacent to \( u \). Since \( x \in B \), it follows that \( x \) is adjacent to a vertex \( w \in V(P_i) \cap B \) for some \( i \in \{1, \ldots, k\} \), say without loss of generality \( w \in V(P_1) \cap B \). Since \( x \) is adjacent to \( u \) and \( w \), \( G \) contains the edge \( uw \). Moreover, \( w \neq u_1 \), since \( u_1 \notin B \). But then replacing the subpath from \( u \) to \( w \) in \( P_1 \), which contains \( u_1 \) in its interior, by the edge \( uw \) yields a shorter \( k \)-tuple of internally vertex-disjoint \( uv \)-paths in \( G \). This contradicts the minimality of \( P_1, \ldots, P_k \).

In particular, a triconnected map graph with outerplanar witness, which is \( K_{2,3} \)-minor free, must be a clique. Hence, it suffices to investigate separators of size 2 in \( G \).

In general map graphs, every two adjacent vertices have at least one neighboring intersection point in the witness, according to the definition of half-squares. Let \( G \) be a map graph with an outerplanar witness \( H \). The intuition for the next lemma is that then every three vertices of a clique (of size at least 4) in \( G \) have a common neighboring intersection point in \( H \). Note that this property is not true for arbitrary planar witnesses, as each of the structures “pizza with crust”, “hamantash” and “riceball” [2] contains three nations without any common intersection; see Fig. 2.

Figure 2: In arbitrary map graphs, cliques can be realized in four possible ways; however, each of the structures “pizza with crust”, “hamantash” and “riceball” contains three nations without any common intersection.

**Lemma 6.** Let \( G \) be a map graph with an outerplanar witness \( W \) and let \( v_0, v_1, v_2 \) be three vertices of a clique of size at least 4. Then there is an intersection point in \( W \) that is adjacent to \( v_0, v_1 \) and \( v_2 \).

**Proof.** Assume that this is not the case. Then there exist three distinct intersection points \( x_0, x_1, x_2 \) such that \( x_i \) is adjacent to \( v_{i+1} \) and \( v_{i+2} \) but not to \( v_i \), where indices are taken modulo 3; see Fig. 3a. Now consider a vertex \( v \) of the clique that is distinct from the \( v_i \). There is a path of length two from \( v \) to each of the \( v_i \) in \( W \). If \( v \) is adjacent to two (or more) of the \( x_i \), we immediately have a \( K_{2,3} \)-minor (Fig. 3b) with branch vertices \( \{x_1, x_2\} \); likewise, if there is an intersection point distinct from the \( x_i \), adjacent to...
two of the \( v_i \) (\( \{v_0, v_2\} \)) in Fig. 3b. It follows that \( v \) must reach one of the \( v_i \), without loss of generality \( v_0 \), via an intersection point \( x \) distinct from the \( x_i \), and \( x_1 \) is not adjacent to \( v_1 \) and \( v_2 \) (Fig. 3b). But now, to reach \( v_1 \) and \( v_2 \), \( v \) either has to be adjacent to \( x_0 \), or it must use a new intersection point \( y \) adjacent to \( v_1 \) or \( v_2 \). In both cases, we obtain a \( K_{2,3} \)-minor.

The proof of Lemma 6 shows that the size bound “4” on the clique size is as small as possible (see Fig. 3a).

Definition 2. A clique \( C \) in a map graph \( G \) with witness \( W \) is represented by an intersection point if \( W \) contains at least one intersection point whose neighborhood is \( V(C) \).

Cliques that are represented by exactly one intersection point are called “pizzas” by Chen et al. [2]. We now show that in outerplanar witnesses all cliques of size at least 4 must be represented in this way, thus significantly reducing the possible representations.

Lemma 7. Let \( G \) be a map graph with outerplanar witness \( W \). Each maximal clique \( C \) of size at least 4 is represented by an intersection point.

Proof. We first show that \( W \) contains an intersection point \( x \) that is adjacent to all vertices of \( C \). This readily implies that \( x \) has no other neighbor, as any such neighbor would contradict the maximality of \( C \). Let \( x \) be an intersection point in \( W \) with a maximum number of neighbors in \( C \) and assume to the contrary that \( C \) contains a vertex \( v_0 \) that is not adjacent to \( x \). By Lemma 6, \( x \) has at least three neighbors \( v_1, v_2, v_3 \) in \( C \). Let \( V' = \{v_0, v_1, v_2, v_3\} \).

If there was a single intersection point \( y \neq x \) adjacent to \( \{v_1, v_2, v_3\} \), this would result in a \( K_{2,3} \) with branch vertices \( x \) and \( y \). This implies that any intersection point different from \( x \) can be adjacent to at most three of the vertices in \( V' \) (omitting at least one of \( \{v_1, v_2, v_3\} \)). On the other hand, by Lemma 6 for any such subset a corresponding intersection point exists. Thus, there exist intersection points \( y \neq x \) and \( z \neq x \) with \( N(y) \cap V' = \{v_0, v_1, v_2\} \) and \( N(z) \cap V' = \{v_0, v_2, v_3\} \); see Fig. 4a. Contracting the two edges \( yv_0 \) and \( zv_0 \) yields a \( K_{2,3} \)-minor in \( W \), contradicting its outerplanarity.

According to the definition of witnesses, maximal cliques of size 2 must also be represented by intersection points. Thus, the only cliques for which the representation is unclear are maximal cliques of size 3. In the following we show that cliques that cannot be represented by an intersection point in an outerplanar witness induce a special structure. Namely, any two of its vertices form a separator, and we further describe the way in which these separators decompose the graph.

Lemma 8. Let \( G \) be a map graph with outerplanar witness and let \( W \) be an outerplanar witness of \( G \) that maximizes the number of maximal cliques of size 3 that are represented by intersection points. Let \( v_0 \) and \( v_1 \) be any two vertices of a maximal clique \( C = \{v_0, v_1, v_2\} \) that is not represented by an intersection point.

\begin{figure}[h]
\centering
\includegraphics{fig4.png}
\caption{Illustration of the proofs of Lemma 7(a) and Lemma 8(b).}
\end{figure}
Then \{v_0, v_1\} is a separator in \(G\) that separates \(v_2\) from every other maximal clique of \(G\) containing \(v_0\) and \(v_1\).

**Proof.** Since \(C\) is not represented by an intersection point, there exist witness points \(x_0, x_1, x_2\) such that \(x_i\) is adjacent to \(v_{i+1}\) and \(v_{i+2}\) but not to \(v_i\) (indices modulo 3). The cycle containing the \(x_i\) and \(v_i\) does not contain any vertex inside because of outerplanarity of \(W\); in addition, any interior edge would contradict that \(C\) is not represented by an intersection point. Thus the \(x_i\) and \(v_i\) form the boundary of a face of \(W\); see Fig. [13].

If some \(x_i\) has degree 2, we can add the edge \(x_i v_1\), which would result in an intersection point for the clique, contradicting the maximality of \(W\). It follows that each \(x_i\) is adjacent to some \(v_i \in V(G) - C\). We claim that \(\{v_0, v_1\}\) separates \(u_2\) from \(v_2\) in \(W\). Otherwise, there exists a path from \(u_2\) to \(v_2\) in \(W - v_0 - v_1\). Together with the cycle formed by the \(v_i\) and \(x_i\), this yields a \(K_{2,3}\)-minor, contradicting the outerplanarity. Thus, \(\{v_0, v_1\}\) is also a separator in \(G\) that separates \(u_2\) and \(v_2\). \(\square\)

### 5.1 Structural Properties of Map Graphs with Outerplanar Witness

To obtain an efficient recognition algorithm for map graphs with outerplanar witness, two things remain to be done. First, we need to better understand the structure of those maximal cliques for which the representation in the witness is not already decided by the previous results. Second, we need to find a way to quickly enumerate all the relevant cliques in order to decide upon their representation in the witness.

As we have seen, the maximal cliques for which the representation cannot be an intersection point induce separating pairs in the input graph. This, together with the fact that certainly all cliques of size at least 4 belong to a single triconnected component of the input graph motivates the study of the triconnected components of the input graph. Essentially, we establish the following connections between the maximal cliques and the triconnected components:

1. Every maximal clique of size at least three shows up as a triconnected component of \(G\).
2. A characterization of the maximal cliques of size three that cannot be represented by an intersection point in terms of triconnected components.

The first property allows us to quickly compute all maximal cliques by exploiting the SPQR-tree, which can be computed in linear time [11], rather then by a maximal clique enumeration algorithm, which might be much slower. The second property is used to determine the correct intersection points for all maximal cliques.

The following corollary follows immediately from applying Lemma [3] along the recursive definition of SPQR-trees.

**Corollary 2.** Let \(G\) be a biconnected map graph with an outerplanar witness. Then each skeleton of the SPQR-tree of \(G\) is a map graph with an outerplanar witness.

Now we derive further structural results on the triconnected components of map graphs with outerplanar witness. The following two lemmas will turn out to be helpful for this, as they exhibit the special structure of separators a map graph with outerplanar witness has.

**Lemma 9.** For a map graph \(G\) with outerplanar witness, the following statements hold:

(i) Any two maximal cliques of \(G\) share at most two vertices.

(ii) Any two vertices that are shared by two maximal cliques of \(G\) form a separator of \(G\) separating these cliques.

(iii) Any two vertices are shared by at most two maximal cliques.

**Proof.** For (i), assume that \(C, C'\) are maximal cliques sharing vertices \(v_1, v_2, v_3\). Since \(C, C'\) are distinct, they have size at least 4. By Lemma [7] they are represented by distinct intersection points \(c, c'\); see Fig. [5a]. Then \(v_1, v_2, v_3, c, c'\) induce a \(K_{2,3}\); a contradiction.

For (ii), let \(\{u, v\}\) be two vertices that are shared by two maximal cliques \(C\) and \(C'\). Consider an outerplanar witness \(W\) of \(G\) that maximizes the number of cliques of size 3 that are represented by intersection points. If \(C\) and \(C'\) are realized by intersection points \(c\) and \(c'\), respectively, consider \(w\) and \(w'\) in \(C \setminus C'\) and \(C' \setminus C\), respectively. A path between these two vertices avoiding \(u\) and \(v\) yields a \(K_{2,3}\)-minor with branch vertices \(c\) and \(c'\), contradicting outerplanarity; see Fig. [6a]. Hence, assume that one of the cliques is not realized as an intersection point. With Lemma [4] this clique has at most three vertices and the statement follows from Lemma [8].
For (iii), assume that two vertices $u$ and $v$ are shared by at least three maximal cliques $C_0, C_1$ and $C_2$. Note that each $C_i$ has size at least three, as otherwise the $C_i$ would not be distinct. According to (ii), every $C_i$ contains a vertex $v_i$ that is not in $C_{i+1} \cup C_{i+2}$ (indices taken modulo 3). If $C_i$ is represented by an intersection point $c_i$, then let $P_i$ denote the path $uc_iv_i$ in $W$ (path $P_i$ in Fig. 5c). If $C_i$ is not represented by an intersection point, then $C_i$ has size 3, and $W$ contains a path $uc_i^1, v_i, c_i^2, v$, where $c_i^1$ and $c_i^2$ are intersection points. We define $P_i$ to be this path (path $P_2$ in Fig. 5c). Paths $P_i$ and $P_j$ for $i \neq j$ are internally disjoint by the definition of the $v_i$, and so three internally disjoint paths from $u$ to $v$ in $W$ yield a $K_{2,3}$-minor. This contradicts the outerplanarity of $W$.

Lemma 10. Let $G$ be a biconnected map graph with an outerplanar witness $W$. Every separator $S = \{u, v\}$ of $G$ of size two separates exactly two components.

Proof. Assume to the contrary that $G - S$ contains at least three components $C_i$, $1 \leq i \leq 3$. According to Lemma 9 the touched subgraphs $W[T(V(C_i))]$ are different components of $W - S$. For every $i$, there is a path $P_i$ in $G$ from $u$ to $v$ that contains a vertex of $C_i$ as inner vertex, since $S$ is minimal in $G$. In $W$, each path $P_i$ corresponds to a path from $u$ to $v$ that contains a vertex of $W[T(V(C_i))]$ as inner vertex (this may be either an intersection point or a real vertex). Since each such path has length at least two, $W$ contains a $K_{2,3}$-minor with branch vertices $u$ and $v$.

The last two structural lemmas allow us to identify restrictions on the 3-connected components.

Lemma 11. Let $G$ be a biconnected map graph with an outerplanar witness. Then the SPQR-tree of $G$ satisfies the following properties:

(i) Every P-node skeleton consists of three parallel edges of which one is a real edge.

(ii) Every R-node skeleton is a clique.

Proof. For (i), observe that, according to Lemma 10 there are exactly two components in $G - S$ for every separator $S = \{u, v\}$ of $G$ of size two. Thus, every parallel P-node in the SPQR-tree has at most three parallel edges (and at least three by definition of SPQR-trees): two virtual ones and one edge from $G$.

For (ii), observe that the skeleton of an R-node is a triconnected graph. According to Corollary 2 this skeleton is a map graph with an outerplanar witness. Applying Lemma 5 with $k = 3$ implies that the skeleton is a clique.

In fact, it turns out that not only every R-node skeleton is a clique, but each such clique is a subgraph of $G$.

Lemma 12. Let $G$ be a map graph with outerplanar witness. Then every R-node skeleton is a maximal clique that is a subgraph of $G$.

Proof. Consider an R-node skeleton $S$, which is a clique by Lemma 11(ii), and let $uv$ be an edge of $S$ that is not in $G$ (thus, a virtual edge). Let $G'$ be the graph that is obtained from $G$ by contracting the subgraph corresponding to each remaining virtual edge into a single edge. Let $G''$ be the subgraph obtained from $G'$ by removing all vertices in the subgraph that corresponds to the virtual edge $uv$, except for $u, v$ and a shortest path between them. The graph $G'$ is a map graph with outerplanar witness by Lemma 8 and $G''$ is a map graph with outerplanar witness by Observation 1 as $G''$ is an induced subgraph of $G'$.

Thus, $G''$ contains the two cliques with vertex sets $V_1 = V(S) - \{u\}$ and $V_2 = V(S) - \{v\}$, which are maximal, as $uv$ is not in $G''$. But then $|V_1 \cap V_2| \geq 2$, since $|V(S)| \geq 4$ and $|V_1 \cap V_2| \leq 2$ by Lemma 9(i).
clique of size 3 in $G$ represented by an intersection point. Then $C$ is not represented by an intersection point in $W$ if and only if each biconnected component $H$ contains at least four vertices, which by Lemma 12 implies a larger clique. Hence, it must be an $S$-node. The skeleton of this $S$-node cannot contain any other vertex than those in $C$, as it then would not be a cycle. 

It follows from Lemma 12 and Lemma 13 that we find all maximal cliques by considering the skeletons of the SPQR-tree. In particular, this allows us to enumerate all maximal cliques in linear time. It remains to understand which maximal cliques of size three may not be represented by an intersection point.

Lemma 14. Let $G$ be a biconnected map graph with outerplanar witness and let $C = \{u, v, w\}$ be a maximal clique in $G$. Then there is an $S$-node skeleton with vertex set $\{u, v, w\}$.

Proof. By definition, the subgraph of $G$ induced by $C$ is triconnected, hence there is a skeleton of a node in the SPQR-tree of $G$ that contains all three vertices of $C$. However, this node cannot be a $P$-node (as it contains only two vertices) and it cannot be an $R$-node, whose skeletons are well-known to contain at least four vertices, which by Lemma 12 implies a larger clique. Hence, it must be an $S$-node. The skeleton of this $S$-node cannot contain any other vertex than those in $C$, as it then would not be a cycle. 

5.2 Recognition Algorithm

Based on our structural observations, we give a linear-time algorithm for recognizing map graphs that admit an outerplanar witness, Algorithm 1. The algorithm takes as input an arbitrary graph $G$. First, it decomposes the graph $G$ into its biconnected components $H_1, \ldots, H_t$. We know that $G$ is a map graph with outerplanar witness if and only if each biconnected component $H_i$ is a map graph with outerplanar witness (see Lemma 2).

We seek to construct a bipartite witness candidate $W$ of $G$ as follows. Let the vertices of $G$ be one side of the bipartition of $W$. For each $H_i$, compute the decomposition into its triconnected components, i.e., its SPQR-tree $T_i$, in linear time 13 11. For each R-node of $T_i$, check whether its vertex set is a clique of $G$. If not, then reject the graph $H_i$ (and hence $G$) as not being a map graph with outerplanar witness. Otherwise, add an intersection point to $W$ that represents that clique.

For each $S$-node of $T_i$ that forms a clique of size 3 and that has a real edge, add an intersection point to $W$ representing the clique.

Finally, for each edge of $G$ that is not yet represented by one of the previously constructed intersection points, add a separate intersection point of degree 2 to $W$ representing exactly this edge. Let $W$ be the resulting candidate witness graph. Test whether $W$ is outerplanar. If $W$ is outerplanar, then output the
Algorithm 1 Linear-Time Recognition Algorithm for Map Graphs with Outerplanar Witness

Input: A graph $G$.
Output: An outerplanar witness $W$ of $G$ if $G$ is a map graph, “no” otherwise.

1. Create a candidate bipartite graph $W$; let $V$ be one side of the bipartition of $W$.
2. for each biconnected component $H_i$ of $G$ do
3.  Compute an SPQR-tree $T_i$ of $H_i$.
4. for each R-node $R$ of $T_i$ do
5.      if the vertices of the skeleton of $R$ do not form a clique in $G$ then
6.          return “no”
7.      Add an intersection point $p_R$ with neighborhood $V(R)$ to $W$.
8. for each S-node $S$ of $T_i$ whose skeleton is a clique of size 3 with some real edge do
9.      Add an intersection point $p_S$ with neighborhood $V(S)$ to $W$.
10. for each edge $e = uv$ in $G$ that is not yet represented by an intersection point do
11.    Add an intersection point $p_{uv}$ of degree 2 with neighborhood $\{u, v\}$ to $W$.
12. Test outerplanarity of $W$: if “yes”, return $W$, else return “no”.

Theorem 3. Map graphs with outerplanar witness can be recognized in $O(n+m)$ time.

Proof. It is not hard to see that the above algorithm can be implemented to run in $O(n+m)$ time. In the following we prove the correctness.

First, assume that the algorithm outputs a witness $W$ in the end. We show that $W$ is a witness of the input graph $G$. Note that all intersection points we create represent either cliques of various sizes of $G$, or they only represent a single edge (if they are added in the last step). Thus, $W^2[V(G)] \subseteq G$. On the other hand, the last step ensures that $W^2[V(G)] \supseteq G$, and thus we have $W^2[V(G)] = G$, which shows that indeed $G$ is a map graph with outerplanar witness $W$.

Conversely, assume that the algorithm rejects $G$, although $G$ is a map graph with outerplanar witness. Let $W^+$ be an outerplanar witness that maximizes the number of cliques of size 3 that are represented by an intersection point.

There are only two steps in the algorithm where $G$ may be rejected. First, when an R-node skeleton is not a clique that is subgraph of $G$. But in this case, $G$ is not a map graph with outerplanar witness by Lemma 12. Second, $G$ may be rejected when $W$ is found not to be outerplanar. In this case, we will give an isomorphism from $W$ to an induced subgraph of $W^+$ whose restriction to $V$ is the identity. This contradicts that $W^+$ is outerplanar.

It suffices to give the mapping for intersection points only, as every witness contains an identical set $V$ of real vertices. Let $w \in W$ be an intersection point of degree at least 4. Then $N(w)$ is a clique of size at least 4 in $G$. The intersection point $w$ was added due to an R-node clique of that size, and hence $N(w)$ is a maximal clique of size at least 4. By Lemma 7 any outerplanar witness contains an intersection point representing that clique; we thus find an image for $w$ in $W^+$. Note that a second vertex with the same neighborhood is not created; this would imply the existence of a second R-node skeleton with the same vertex set, which is impossible since any two skeletons share at most two vertices.

Let $w \in W$ be an intersection point of degree 3. Then $N(w)$ is a maximal clique of size 3, and $w$ was created due to an S-node skeleton that contained a real edge. By Lemma 14, $N(w)$ is represented by an intersection point in $W^+$ as well, and we thus find an image of $w$ in $W^+$. Again, any intersection point mapped to the same image would imply the existence of a second S-node skeleton with the same vertex set, which is not possible.

Finally, let $w \in W$ be an intersection point of degree 2, which must have been added in the last step, and let $N(w) = \{u, v\}$. Clearly, $W^+$ must contain an intersection point $x$ adjacent to $u$ and $v$. If $x$ has degree at least 4, then $N(x)$ is a clique of size at least 4, which must show up as an R-node of the SPQR-tree. But then the edge $uv$ was already represented in $W$ by an intersection point corresponding to $x$ and the algorithm would not have added $w$. Similarly, if $x$ has degree 3, then $N(x)$ is a maximal 3-clique. Since it is represented by the intersection point $x$, it follows that the corresponding S-node contains a real edge by Lemma 14. But then, again, the algorithm would have added an intersection point to $W$ that corresponds to $x$.
Thus, the degree of $x$ must be 2, and we can choose it as an image for $w$. Since all degree-2 intersection points inserted in the last step represent distinct edges of $G$, no two degree-2 intersection points of $W$ are mapped to the same intersection point of $W^*$. Hence, we have found an isomorphism from $W$ to a subgraph of $W^*$.

The correctness proof shows that the algorithm computes a smallest (with respect to subgraph inclusion) outerplanar witness, and that this witness is unique up to isomorphism.

6 Discussion

We gave an $O(n + m)$ time and space recognition algorithm for map graphs with an outerplanar witness. The algorithm is certifying. This result is a first step towards improving Thorup’s recognition algorithm for map graphs with planar witness that requires time about $\Omega(n^{1.2})$.

For map graphs with outerplanar witness, we proved that that every two maximal cliques of $G$ intersect in at most two vertices (Lemma 9). One might hope that for arbitrary map graphs at least the intersection of $k$ maximal cliques (for some $k \geq 2$) is bounded from above. The following infinite graph class however shows that this is not the case:

![Figure 6: Witness of a map graph that has many cliques that all share a large set of vertices.](image)

Lemma 15. For every pair of positive integers $k$ and $\ell$, there exist map graphs $G_{k,\ell}$ with exactly $k$ maximal cliques such that these cliques intersect in exactly $\ell$ vertices.

Proof. We describe a witness for $G_{k,\ell}$. It consists of a set $A$ of $\ell$ vertices, two intersection points $x$ and $y$ that are both adjacent to all vertices in $A$, and, for every $i = 1, \ldots, k$, the real vertices $a_i, b_i$, an intersection point $x_i$, and the edges $xa_i, a_ix_i, x_ib_i$ and $b_iy$ (see Fig. 6). Since $a_i$ and $b_j$ are adjacent in $G_{k,\ell}$ if and only if $i = j$, each of the vertex sets $A \cup \{a_i, b_i\}, i = 1, \ldots, k$ is a maximal clique of $G_{k,\ell}$. The maximal cliques thus intersect in exactly the $\ell$ vertices of $A$.

References


